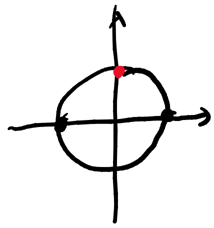
$w = 1 + 2i$

$(w-1)^6 = (2i) = 2^6 i^6 = 64(i^2)^3 = 64(-1)^3 = -64$

Oppure, usando la forma trigonometrica:



$w - 1 = 2i$

$|2i| = \sqrt{0+4} = 2$

$\cos\theta = 0 \wedge \sin\theta = \frac{2}{2} = 1$

$\theta = \frac{\pi}{2}$

$(w-1)^6 = 2^6\left(\cos(3\pi) + i\sin(3\pi)\right) = 64(-1+0\cdot i) = -64$

Ricordare:

$$\cos(\arg(w)) = \frac{Re(w)}{|w|} \wedge \sin(\arg(w)) = \frac{Im(w)}{|w|}$$

cioè $w = x + iy$

$$\cos(\arg(w)) = \frac{x}{\sqrt{x^2+y^2}} \wedge \sin(\arg(w)) = \frac{y}{\sqrt{x^2+y^2}}$$

Studio di funzioni

1) Dominio
2) Determinare eventuali simmetrie.
3) Intersezioni con gli assi (se possibile)
4) Segno della funzione (se possibile)
5) Limiti agli estremi del dominio / asintoti.
6) Derivata.
7) Zeri e il segno della derivata
8) Grafico.
9) Altro: convessità / immagine / massimi e minimi

Principali regole per i domini

- Denominatore $\neq 0$
- Argomento delle radici (di indice pari) deve essere ≥ 0
- Argomento dei logaritmi > 0.

ESEMPI

1) $f(x) = \dfrac{e^{\frac{1}{x^2-2}}}{x+3}$

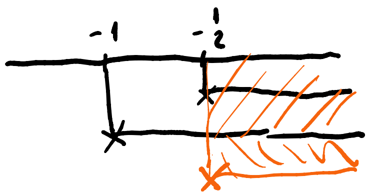
condizioni: $\begin{cases} x+3 \neq 0 & \rightarrow \quad x \neq -3 \\ x^2-2 \neq 0 & \rightarrow \quad x \neq \sqrt{2},\ x \neq -\sqrt{2} \end{cases}$

$x^2 - 2 = 0 \iff x^2 = 2 \iff x = \pm\sqrt{2}$

$\mathrm{Dom}(f) = \mathbb{R} \setminus \{-3, -\sqrt{2}, \sqrt{2}\} = (-\infty, -3) \cup (-3, -\sqrt{2}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{2}, +\infty)$

2) $f(x) = \log(2x+1) - \log(x+1)$

$\begin{cases} 2x+1 > 0 \\ x+1 > 0 \end{cases} \iff \begin{cases} x > -\frac{1}{2} \\ x > -1 \end{cases}$



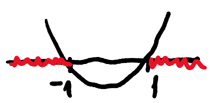
$x > -\frac{1}{2}$

$\mathrm{Dom}(f) = (-\frac{1}{2}, +\infty)$

3) $f(x) = \dfrac{x^2}{e^x \sqrt{x^2-1}}$

$\begin{cases} x^2 - 1 \geq 0 \\ e^x \sqrt{x^2-1} \neq 0 \end{cases}$

- $x^2 - 1 \geq 0 \qquad (x^2-1 = 0 \iff x^2 = 1 \iff x = \pm 1)$

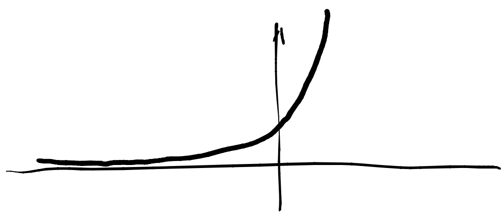


  $x \geq 1 \ \vee \ x \leq -1$

- $e^x \sqrt{x^2-1} = 0$

  $e^x = 0 \quad \vee \quad \sqrt{x^2-1} = 0$

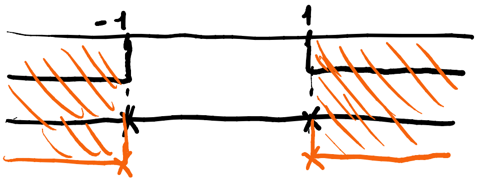
  $\nexists\, x \in \mathbb{R}$ t.c. $e^x = 0$



$\sqrt{x^2-1} = 0 \qquad x^2-1=0 \iff x = \pm 1$

Quindi $\quad e^x\sqrt{x^2-1} \neq 0 \iff x\neq -1,\ x\neq 1$

$$\begin{cases} x\geq 1 \quad \vee \quad x\leq -1 \\ x\neq -1,\ x\neq 1 \end{cases} \iff x>1 \quad \vee \quad x<-1$$



$\mathrm{Dom}(f) = (-\infty,-1)\cup(1,+\infty)$

4) $f(x) = \dfrac{e^x}{x^2+4}$

$x^2+4\neq 0$ sempre vera $\qquad (x^2+4=0$ non ha soluzioni$)$

Quindi $\mathrm{Dom}(f) = \mathbb{R}$.

Derivate :

- Derivate delle funzioni elementari:
- Regole di derivazione :

$$(f+g)' = f' + g'$$
$$(f-g)' = f' - g'$$
$$(fg)' = f'g + fg'$$
$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$
$$(g(f(x)))' = g'(f(x))\cdot f'(x)$$

$$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

ESEMPI

$$\left(e^{\sin x}\right)' = e^{\sin x}\cdot(\sin x)' = e^{\sin x}\cos x$$

$$\left(\sqrt{1+2x}\right)' = \frac{1}{2\sqrt{1+2x}}\cdot(1+2x)' = \frac{1}{\cancel{2}\sqrt{1+2x}}\cdot\cancel{2} = \frac{1}{\sqrt{1+2x}}$$

$$\left(\frac{e^{2x}}{x^2-1}\right)' = \frac{(e^{2x})'(x^2-1) - e^{2x}(x^2-1)'}{(x^2-1)^2}$$

$$= \frac{e^{2x} \cdot 2(x^2-1) - e^{2x}\, 2x}{(x^2-1)^2}$$

$$= \frac{2e^{2x}(x^2-1-x)}{(x^2-1)^2}$$

$$\left(\frac{\sqrt{x^2+2}}{x^2+x+1}\right)' = \frac{(\sqrt{x^2+2})' \cdot (x^2+x+1) - \sqrt{x^2+2}\,(2x+1)}{(x^2+x+1)^2}$$

$$= \frac{\frac{1}{\cancel{2}\sqrt{x^2+2}} \cdot \cancel{2}x \cdot (x^2+x+1) - \sqrt{x^2+2}\,(2x+1)}{(x^2+x+1)^2}$$

$$= \frac{\frac{x(x^2+x+1)}{\sqrt{x^2+2}} - \sqrt{x^2+2}\,(2x+1)}{(x^2+x+1)^2}$$

$$= \frac{\frac{x(x^2+x+1) - (x^2+2)(2x+1)}{\sqrt{x^2+2}}}{(x^2+x+1)^2}$$

$$= \frac{x^3 + \cancel{x^2} + x - 2x^3 - \cancel{x^2} - 4x - 2}{\sqrt{x^2+2}\,(x^2+x+1)^2}$$

$$= \frac{-2x^3 - 3x - 2}{\sqrt{x^2+2}\,(x^2+x+1)^2} = -\frac{2x^3+3x+2}{\sqrt{x^2+2}\,(x^2+x+1)^2}$$

- $f(x) = \log(2x^4 + x^2 + 1)$

  $$f'(x) = \frac{1}{2x^4+x^2+1} \cdot (8x^3 + 2x) = \frac{2x(4x^2+1)}{2x^4+x^2+1}$$

- $f(x) = x^2\sqrt{x^3+1}$

  $$f'(x) = 2x\sqrt{x^3+1} + x^2(\sqrt{x^3+1})'$$

  $$= 2x\sqrt{x^3+1} + x^2 \frac{1}{2\sqrt{x^3+1}} \cdot 3x^2$$

$$= 2x\sqrt{x^3+1} + \frac{3x^4}{2\sqrt{x^3+1}}$$

$$= \frac{2x \cdot 2(x^3+1) + 3x^4}{2\sqrt{x^3+1}} = \frac{4x^4 + 4x + 3x^4}{2\sqrt{x^3+1}}$$

$$= \frac{7x^4 + 4x}{2\sqrt{x^3+1}}$$

Limiti:

Forme indeterminate:

$+\infty - \infty \quad , \quad -\infty + \infty$

$0 \cdot \pm\infty \quad , \quad \pm\infty \cdot 0$

$\frac{\pm\infty}{\pm\infty} \quad \frac{0}{0}$] Teorema di De l'Hopital

$1^{\pm\infty} \quad 0^0 \quad (+\infty)^0$

$$\lim_{x\to+\infty} x^2 - x + 1 = \lim_{x\to+\infty} \overset{\to+\infty}{x^2}\left(1 - \overset{\to 0}{\frac{1}{x}} + \overset{\to 0}{\frac{1}{x^2}}\right) = +\infty \cdot 1 = +\infty$$

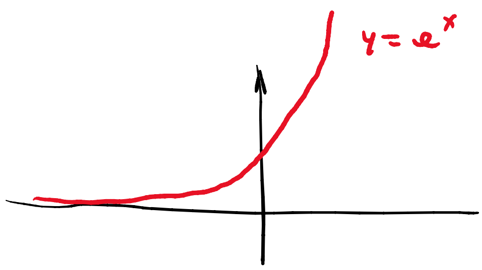
$$\lim_{x\to+\infty} x^2 - 3x^3 = \lim_{x\to+\infty} \overset{\to+\infty}{x^3}\left(\overset{\to 0}{\frac{1}{x}} - 3\right) = +\infty \cdot (-3) = -\infty .$$

$$\lim_{x\to+\infty} \frac{x^2+x^3-2}{3x^3+4} = \lim_{x\to+\infty} \frac{\cancel{x^3}}{3\cancel{x^3}} = \frac{1}{3}$$

- $\displaystyle\lim_{x\to+\infty} \frac{\sqrt{x^2+1}}{1-x} = \lim_{x\to+\infty} \frac{\sqrt{x^2}}{-x} = \lim_{x\to+\infty} \frac{x}{-x} = \lim_{x\to+\infty} -\frac{x}{x} = -1$

- $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{1-x} = \lim_{x\to-\infty} \frac{\sqrt{x^2}}{-x} = \lim_{x\to-\infty} \frac{|x|}{-x} = \lim_{x\to-\infty} \frac{-x}{-x} = 1$

- $\displaystyle\lim_{x\to+\infty} e^{\frac{1}{x}} = e^0 = 1$



- $\displaystyle\lim_{x\to+\infty} e^{x-x^2} = e^{-\infty} = 0$

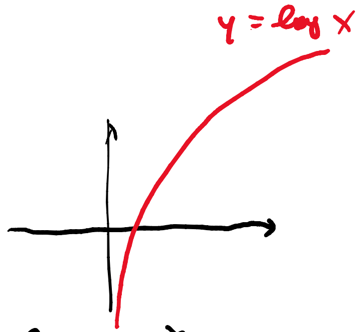
Per le forme indeterminate del tipo $\frac{\pm\infty}{\pm\infty}$ o $\frac{0}{0}$ si può usare anche il teorema di De l'Hopital:

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{f'(x)}{g'(x)}$$

ESEMPI

- $\lim_{x \to +\infty} \frac{e^x}{3x+1}$ $\frac{+\infty}{+\infty}$

$$= \lim_{x \to +\infty} \frac{e^x}{3} = \frac{+\infty}{3} = +\infty$$



- $\lim_{x \to +\infty} \frac{\log(x^2+1)}{x}$ $\frac{+\infty}{+\infty}$

$$\lim_{x \to +\infty} \frac{\frac{1}{x^2+1} \cdot 2x}{1} = \lim_{x \to +\infty} \frac{x}{x^2+1} = \lim_{x \to +\infty} \frac{x}{x^2} = \lim_{x \to +\infty} \frac{1}{x} = 0$$

Nota:

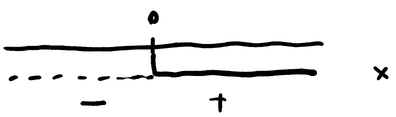
Nei limiti attenzione ai risultati del tipo $\frac{c}{0}$ con $c \neq 0$.

Il risultato può essere $+\infty$, $-\infty$ oppure può essere necessario distinguere il limite destro dal limite sinistro.

Bisogna capire qual è il segno del denominatore.

- $\lim_{x \to 0} \frac{1}{x^2} = \frac{1}{0^+} = +\infty$

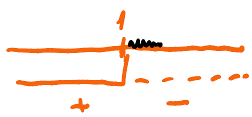
- $\lim_{x \to 0} \frac{1}{x}$ bisogna distinguere



limite destro e limite sinistro:

$$\lim_{x \to 0^+} \frac{1}{x} = \frac{1}{0^+} = +\infty \quad e \quad \lim_{x \to 0^-} \frac{1}{0^-} = -\infty$$

- $\lim_{x \to 1} \frac{x}{1-x}$ $\frac{1}{0}$ $1 - x > 0 \iff x < 1$



$$\lim_{x \to 1^+} \frac{x}{1-x} = \frac{1}{0^-} = -\infty \qquad \lim_{x \to 1^-} \frac{x}{1-x} = \frac{1}{0^+} = +\infty.$$

ESEMPIO DI STUDIO DI FUNZIONE

$$f(x) = \frac{x^4}{x-2}$$

1) Dom(f) : $x - 2 \neq 0 \iff x \neq 2$

$\text{Dom}(f) = \mathbb{R} \setminus \{2\} = (-\infty, 2) \cup (2, +\infty)$

2) $f(-x) = \dfrac{(-x)^4}{-x-2} = -\dfrac{x^4}{x+2} \neq f(x)$ e $\neq -f(x)$

f non è pari né dispari.

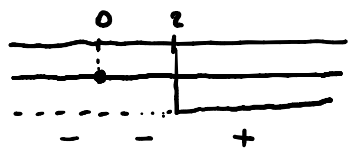
3) Asse y: $f(0) = 0$ $(0,0)$ è intersezione del grafico con l'asse y.

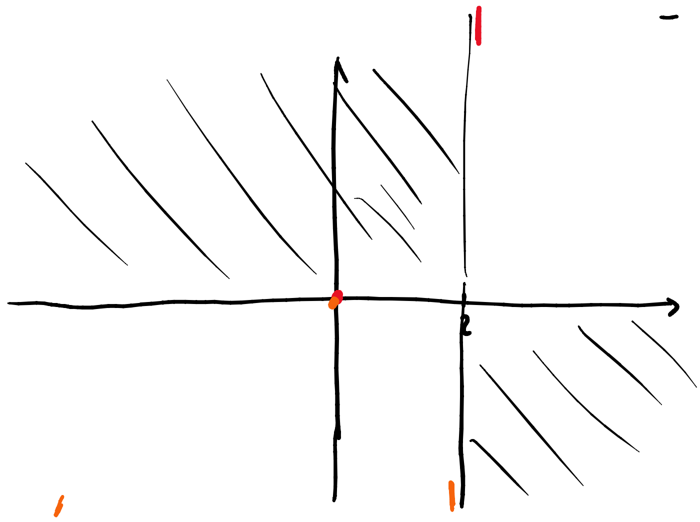
Asse x: $f(x) = 0 \iff \dfrac{x^4}{x-2} = 0 \iff x^4 = 0 \iff 0$

$(0,0)$ è anche l'unica intersezione del grafico con l'asse x.

4) Segno: $\dfrac{x^4}{x-2} > 0$

- $x^4 \geq 0 \quad \forall x \in \mathbb{R}$
- $x - 2 > 0 \iff x > 2$





5) limiti :

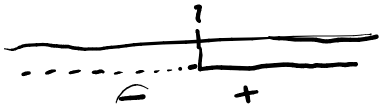
$$\lim_{x \to +\infty} \frac{x^4}{x-2} = \lim_{x \to +\infty} \frac{x^4}{x} = \lim_{x \to +\infty} x^3 = +\infty .$$

$$\lim_{x \to -\infty} \frac{x^4}{x-2} = \lim_{x \to -\infty} \frac{x^4}{x^3} = \lim_{x \to -\infty} x^3 = -\infty$$

$$\lim_{x\to 2} \frac{x^4}{x-2} \qquad \frac{16}{0}$$

$$\lim_{x\to 2^+} \frac{x^4}{x-2} = \frac{16}{0^+} = +\infty$$

$$\lim_{x\to 2^-} \frac{x^2}{x-2} = \frac{16}{0^-} = -\infty$$



6) $$f'(x) = \left(\frac{x^4}{x-2}\right)' = \frac{4x^3(x-2) - x^4\cdot 1}{(x-2)^2}$$

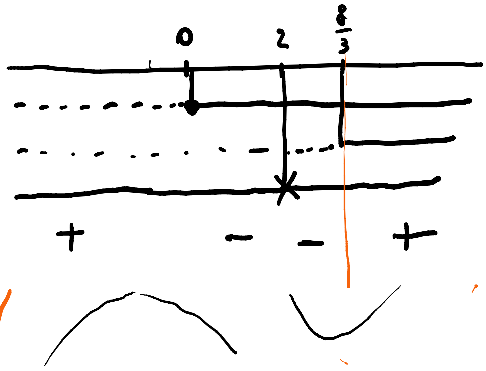
$$= \frac{4x^4 - 8x^3 - x^4}{(x-2)^2}$$

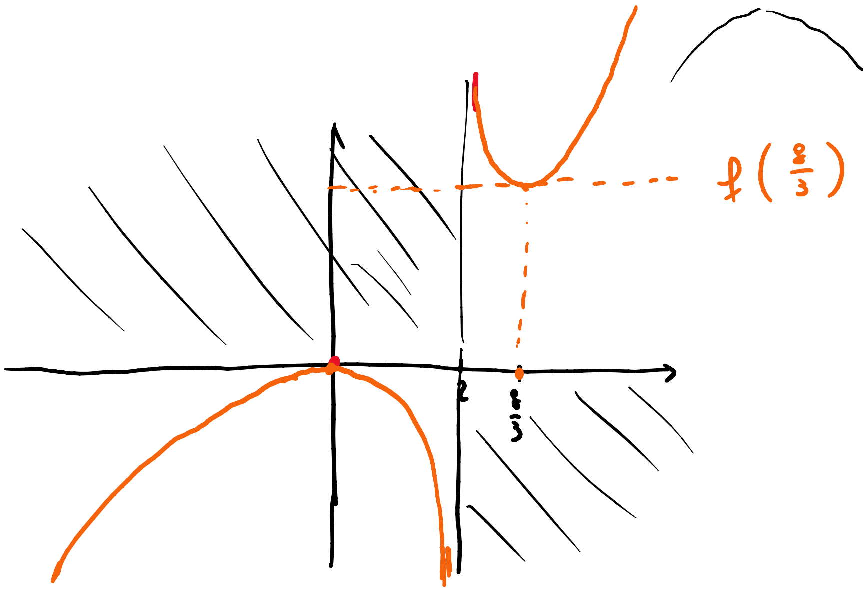
$$= \frac{3x^4 - 8x^3}{(x-2)^2} = \frac{x^3(3x-8)}{(x-2)^2}$$

7) Segno della derivata:

$$\frac{x^3(3x-8)}{(x-2)^2} > 0$$

- $x^3 > 0 \iff x > 0$
- $3x - 8 > 0 \iff x > \frac{8}{3}$
- $(x-2)^2 > 0 \iff \forall\, x \in \mathbb{R}\setminus\{2\}$





ESEMPIO 2

$$f(x) = \frac{e^{x^2}}{3-x^2}$$

1) Dominio: $3 - x^2 \neq 0$. $\quad x^2 - 3 = 0 \iff x = \pm\sqrt{3}$

$$\mathrm{Dom}(f) = \mathbb{R} \setminus \{-\sqrt{3}, +\sqrt{3}\}$$

2) Simmetrie: $f(-x) = \dfrac{e^{(-x)^2}}{3-(-x)^2} = \dfrac{e^{x^2}}{3-x^2} = f(x)$

f è pari.

3) Int. con gli assi:

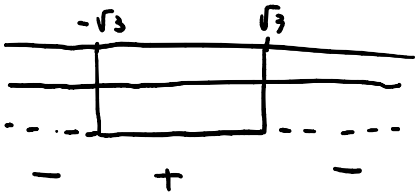
Asse y: $f(0) = \dfrac{e^0}{3-0} = \dfrac{1}{3}$ $\qquad$ $(0, \frac{1}{3})$ è l'intersezione del grafico con l'asse y

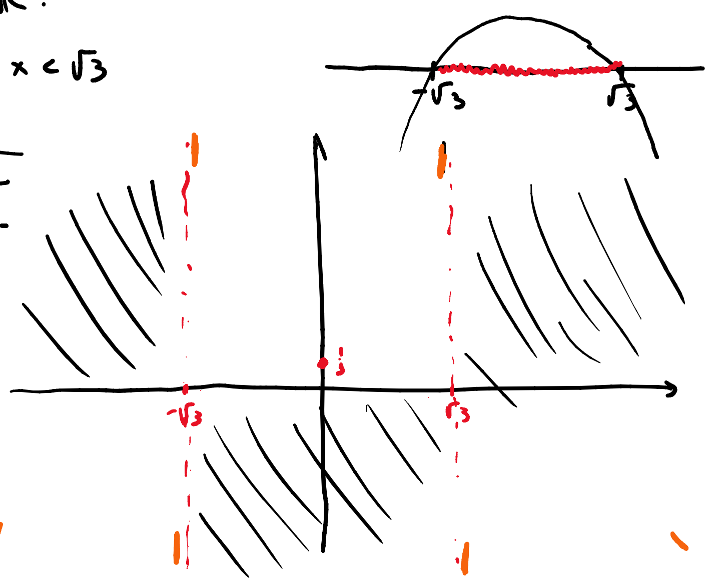
Asse x: $f(x) = 0 \iff \dfrac{e^{x^2}}{3-x^2} = 0 \iff e^{x^2} = 0$ impossibile

Non ci sono intersezioni del grafico con l'asse x.

4) $\dfrac{e^{x^2}}{3-x^2} > 0$

- $e^{x^2} > 0 \quad \forall x \in \mathbb{R}$.
- $3 - x^2 > 0$, $\quad -\sqrt{3} < x < \sqrt{3}$



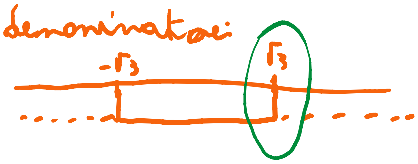


5) limiti

- $\displaystyle\lim_{x \to +\infty} \frac{e^{x^2}}{3-x^2}$ $\qquad \dfrac{+\infty}{-\infty}$

$$= \lim_{x \to +\infty} \frac{e^{x^2} \cdot 2x}{-2x} = \lim_{x \to +\infty} -e^{x^2} = -\infty.$$

• $\lim_{x \to \sqrt{3}} \frac{e^{x^2}}{3-x^2}$ $\frac{e^3}{0}$ segno del denominatore:



$$\lim_{x \to (\sqrt{3})^+} \frac{e^{x^2}}{3-x^2} = \frac{e^3}{0^-} = -\infty$$

$$\lim_{x \to (\sqrt{3})^-} \frac{e^{x^2}}{3-x^2} = \frac{e^3}{0^+} = +\infty$$

Siccome f è pari:

$$\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = -\infty$$

$$\lim_{x \to (-\sqrt{3})^-} f(x) = \lim_{x \to \sqrt{3}^+} f(x) = -\infty$$

$$\lim_{x \to (-\sqrt{3})^+} f(x) = \lim_{x \to \sqrt{3}^-} f(x) = +\infty$$

6) Derivata

$$f'(x) = \left(\frac{e^{x^2}}{3-x^2}\right)' = \frac{e^{x^2} \cdot 2x\,(3-x^2) - e^{x^2}(-2x)}{(3-x^2)^2}$$

$$= \frac{e^{x^2}\, 2x\,(3-x^2) + e^{x^2} \cdot 2x}{(3-x^2)^2}$$

$$= \frac{e^{x^2} \cdot 2x\,(3-x^2+1)}{(3-x^2)^2} = \frac{e^{x^2} \cdot 2x\,(4-x^2)}{(3-x^2)^2}$$

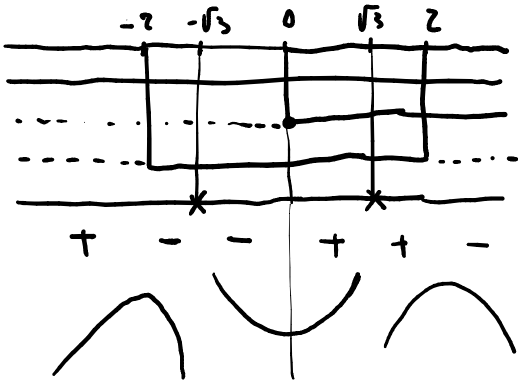
a) Segno di f':

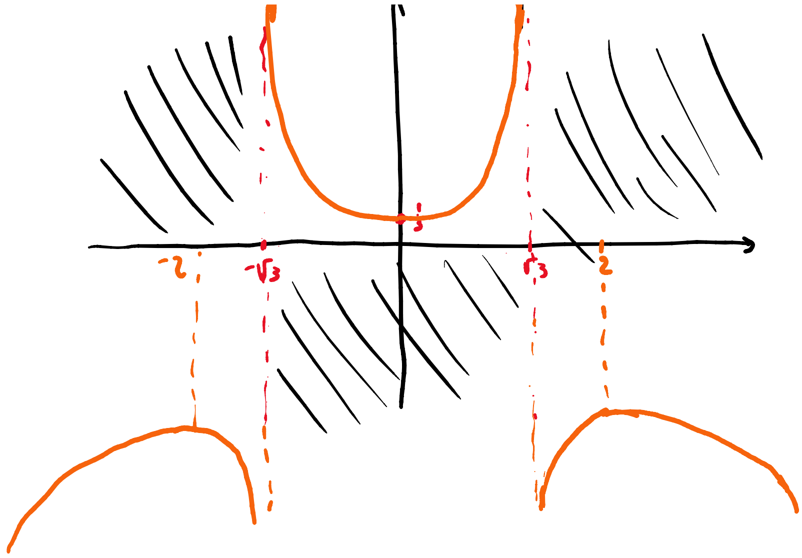
$e^{x^2} > 0 \quad \forall x \in \mathbb{R}$

$2x > 0 \iff x > 0$

$4 - x^2 > 0 \iff -2 < x < 2$

$(3-x^2)^2 > 0 \quad \forall x \in \mathbb{R} \setminus \{\pm\sqrt{3}\}$





$$f(2) = \frac{e^4}{-1} = -e^4$$

ESERCIZI

- $f(x) = \dfrac{e^{2x}}{x-1}$
- $f(x) = \log\left(\dfrac{3x}{x^2+2}\right)$
- $f(x) = \dfrac{\sqrt{x}}{3x^2+1}$
- $f(x) = \dfrac{x}{\sqrt{4x^4+1}}$